

math

we consider m asset on n periods.

portfolio: $\mathbf{b} = [b_1, \dots, b_m]^\top$, where b_i denotes the proportion of wealth invested on asset i .

relative price vector on period i : $\mathbf{x}_i = [x_{i,1}, \dots, x_{i,m}]^\top$, where $x_{i,j}$ denotes the relative price of asset j on period i .

price matrix $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_n] \in \mathbb{R}^{m \times n}$.

mean of returns on periods:

$$\mu_b = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i^\top \mathbf{b} = \bar{\mathbf{x}}^\top \mathbf{b},$$
$$\text{where } \bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i.$$

variance of returns on periods:

$$\begin{aligned}\sigma_b^2 &= E((\mathbf{x}_i^\top \mathbf{b})^2) - [E(\mathbf{x}_i^\top \mathbf{b})]^2 \\ &= \frac{1}{n} \sum_{i=1}^n (\mathbf{x}_i^\top \mathbf{b})^2 - (\bar{\mathbf{x}}^\top \mathbf{b})^2 \\ &= \frac{1}{n} \sum_{i=1}^n \mathbf{b}^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{b} - \mathbf{b}^\top \bar{\mathbf{x}} \bar{\mathbf{x}}^\top \mathbf{b} \\ &= \mathbf{b}^\top \left(\frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i^\top - \bar{\mathbf{x}} \bar{\mathbf{x}}^\top \right) \mathbf{b} \\ &= \mathbf{b}^\top \Sigma \mathbf{b}, \\ \text{where } \Sigma &= \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i^\top - \bar{\mathbf{x}} \bar{\mathbf{x}}^\top = \frac{1}{n} \mathbf{X} \mathbf{X}^\top - \bar{\mathbf{x}} \bar{\mathbf{x}}^\top.\end{aligned}$$

we formulate it as an optimization problem:

$$\begin{aligned} & \textit{minimize} \quad \lambda \mathbf{b}^\top \Sigma \mathbf{b} - \bar{\mathbf{x}}^\top \mathbf{b} \\ & \textit{s.t.} \quad \mathbf{b}^\top \mathbf{1} = 1, \\ & \quad \quad \mathbf{b} \succeq \mathbf{0} \end{aligned}$$

where λ is the risk-aversion parameter.